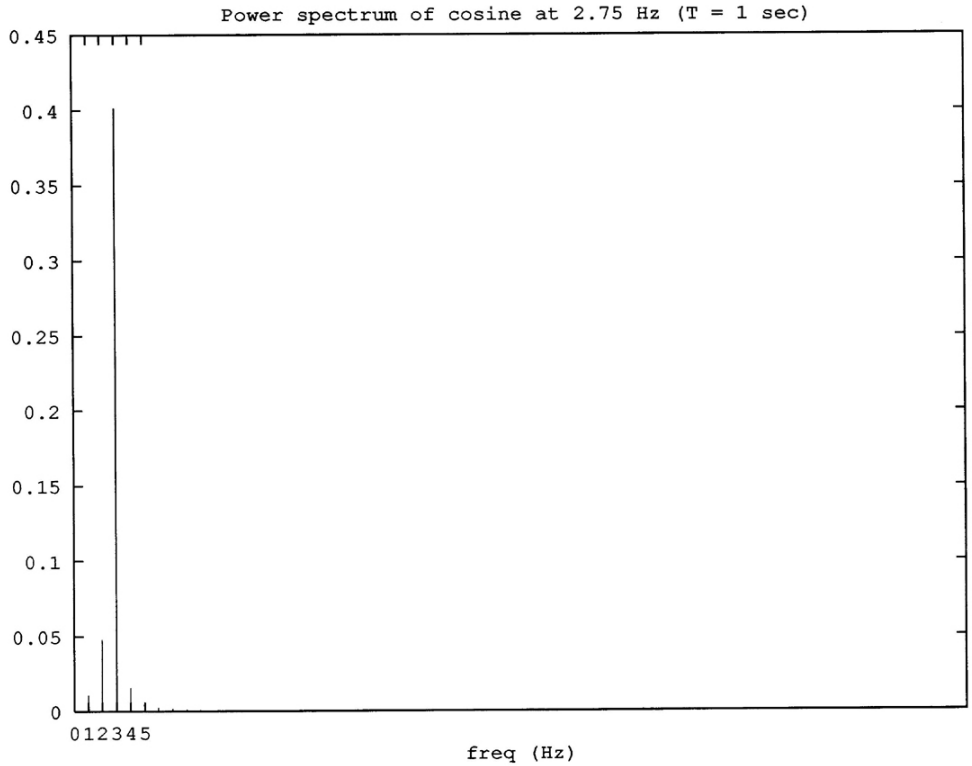
VIII. Spectral Leakage

The power spectrum represents the average distribution of power of a time series as a set of components harmonically related to (integer multiples of) the fundamental frequency. When all the power of the time series occurs at frequencies that are integer multiples of the fundamental frequency, then it can be adequately represented by the Fourier series.

However, a problem arises when there is power in the time series at frequency components that are not harmonics of the fundamental. When this happens, the power of these components is misrepresented. <u>Spectral leakage</u> refers to the misrepresentation of components other than integer multiples of the fundamental frequency.



Consider a unit cosine at frequency ω (2.75 Hz) that lies just below some harmonic of ω_0, e.g. $3\omega_0$ (3 Hz; T=1 sec; ω_0=1 Hz). The DFT decomposes the time series into frequency components at all values of n. None of these corresponds to the cosine frequency, ω, but the coefficients are largest at n=3 and next largest at n=2. There is a gradual diminution of component amplitudes as n departs from these values.

Note that 80% of the total power (0.5) is at n=3.

•The power of the cosine is dispersed or "leaked" out from its true frequency component ω into the components of the Fourier series representation. The reason for the misrepresentation is that only a finite length of the time function has been used for analysis. If we could take an arbitrarily long time sample, we could in principle represent any frequency exactly by finding a fundamental frequency that was harmonically related to it.

•Note that no spurious power is added by leakage since the total power of the Fourier series components equals the total power of the time function (0.5 for the cosine). The frequency components that are closest to the true frequencies in the time series have the greatest power, but power is also distributed over more remote frequencies.

<u>Interpretation of the power spectrum requires care:</u>
The true frequencies may possibly be deduced from those components with the greatest power, but power might also be interpreted as being present at certain frequencies where there actually is none.

Leakage is a particular problem when a large amount of power is present at a single frequency that is non-harmonically related to the fundamental frequency. Because of the high power, appreciable leakage can extend over a wide range of frequencies. This can occur, for example, if there is an artifact at a single frequency in the time series such as the 60 Hz line frequency.

<u>To see how leakage occurs:</u>

As in our example, consider a time series that is a unit cosine wave of arbitrary finite length (T) and frequency ω, and is not an integer multiple of the fundamental frequency. We know that its power is 0.5. We wish to determine how this power is distributed across Fourier series components at $n\omega_0$.

First we compute the real coefficients as:

$$A(n) = \frac{2}{T}\int_0^T X(t)\cos(n\omega_0 t)dt = \frac{2}{T}\int_0^T \cos(\omega t)\cos(n\omega_0 t)dt$$

$$Since \int \cos(at)\cos(bt)dt = \frac{\sin(a-b)t}{2(a-b)} + \frac{\sin(a+b)t}{2(a+b)}$$

$$A(n) = \frac{2}{T}\left[\frac{\sin(\omega - n\omega_0)t}{2(\omega - n\omega_0)} + \frac{\sin(\omega + n\omega_0)t}{2(\omega + n\omega_0)}\right]\Bigg|_0^T$$

$$= \frac{1}{T}\left[\frac{\sin[T(\omega - n\omega_0)]}{(\omega - n\omega_0)} + \frac{\sin[T(\omega + n\omega_0)]}{(\omega + n\omega_0)}\right]$$

And we compute the imaginary coefficients as:

$$B(n) = \frac{2}{T}\int_0^T X(t)\sin(n\omega_0 t)dt = \frac{2}{T}\int_0^T \cos(\omega t)\sin(n\omega_0 t)dt$$

$$Since \int \cos(at)\sin(bt)dt = \frac{\cos(a+b)t}{2(a+b)} - \frac{\cos(a-b)t}{2(a-b)}$$

$$B(n) = \frac{2}{T}\left[\frac{\cos(\omega+n\omega_0)t}{2(\omega+n\omega_0)} - \frac{\cos(\omega-n\omega_0)t}{2(\omega-n\omega_0)}\right]\Bigg|_0^T$$

$$= \frac{1}{T}\left[\frac{\cos[T(\omega+n\omega_0)] - 1}{(\omega+n\omega_0)} - \frac{\cos[T(\omega-n\omega_0)] - 1}{(\omega-n\omega_0)}\right]$$

If we consider that ω is large enough that the denominators will be much larger than the numerators, then we may neglect the term in each equation for which the denominator is $(\omega + n\omega_0)$. Then:

$$A(n) \cong \frac{1}{T} \frac{\sin T(\omega - n\omega_0)}{(\omega - n\omega_0)}$$

$$B(n) \cong \frac{1}{T} \frac{1 - \cos[T(\omega - n\omega_0)]}{(\omega - n\omega_0)}$$

Then the total power at frequency $n\omega_0$ is:

$$Power\,(n) = |Z(n)|^2 + |Z(-n)|^2 = \frac{1}{2}[\,A(n)^2 + B(n)^2\,]$$

$$\cong \frac{1}{2}\frac{\sin^2[T(\omega - n\omega_0)] + (1 - \cos[T(\omega - n\omega_0)])^2}{[T(\omega - n\omega_0)]^2}$$

$$= \frac{1}{2}\frac{\sin^2[T(\omega - n\omega_0)] + 1 - 2\cos[T(\omega - n\omega_0)] + \cos^2[T(\omega - n\omega_0)]}{[T(\omega - n\omega_0)]^2}$$

$$Since\ \sin^2(x) + \cos^2(x) = 1,\ Power\,(n) = \frac{1}{2}\frac{2(1 - \cos[T(\omega - n\omega_0)])}{[T(\omega - n\omega_0)]^2}$$

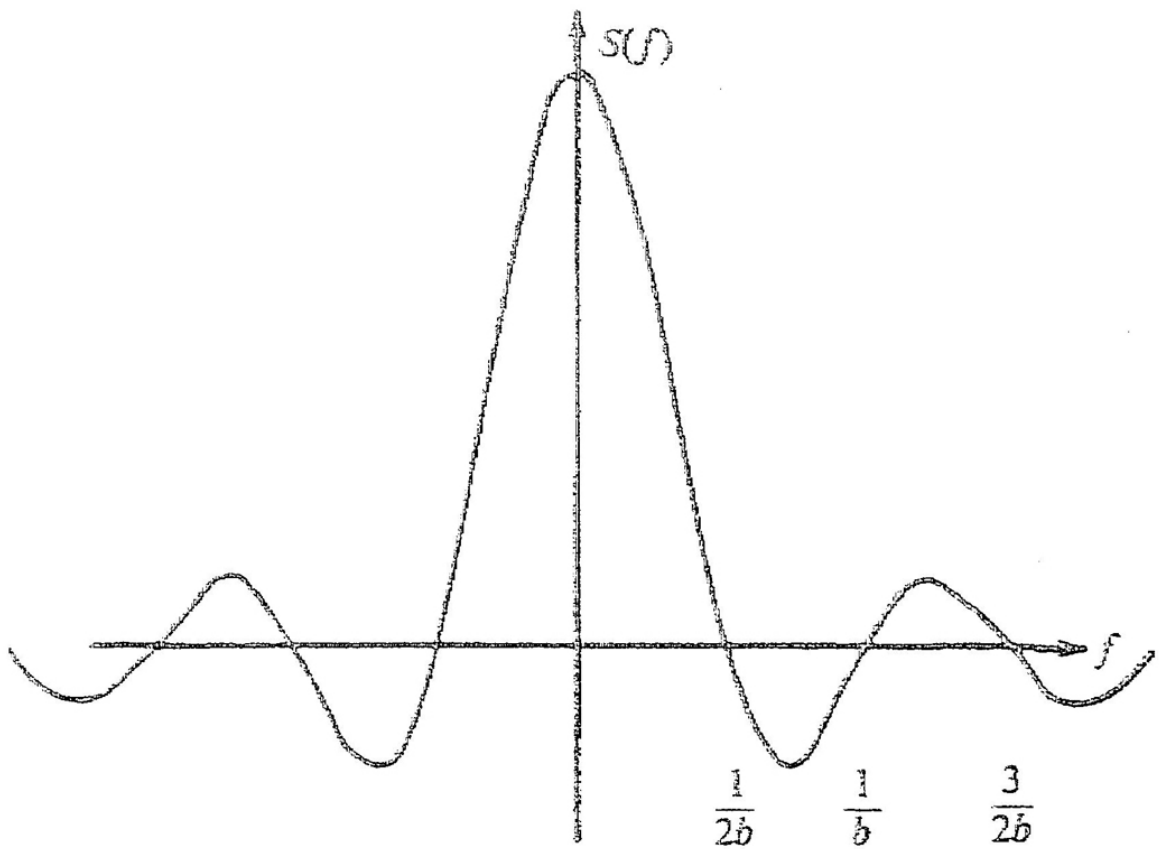
$$\cos\left(x\right) = \cos\left(\frac{x}{2} + \frac{x}{2}\right) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$\therefore \textit{Power}\ (n) = \frac{1 - \left[\cos^2\frac{T}{2}(\omega - n\omega_0) - \sin^2\frac{T}{2}(\omega - n\omega_0)\right]}{\left[T(\omega - n\omega_0)\right]^2}$$

Since 1 - $\cos^2(x)$ = $\sin^2(x)$,

$$\textbf{\textit{Power (n)}} = \frac{2\sin^2\left[\frac{T}{2}(\omega - n\omega_0)\right]}{\left[T(\omega - n\omega_0)\right]^2} = \frac{1}{2}\left[\frac{\sin\frac{T}{2}(\omega - n\omega_0)}{\frac{T}{2}(\omega - n\omega_0)}\right]^2$$

The bracketed term has the form of a **sinc function** [sin(x)/x].



The sinc function is a tapered cosine function. It decreases with increasing x.

Summary to this point

1. The total power of $x(t)=\cos(\omega t)$ is concentrated solely at frequency ω.
2. This power is represented by the DFT at frequencies that are harmonically related to ω_0.
3. The spectral component at any frequency, $n\omega_0$, includes a leaked contribution from frequency ω. The sinc^2 function determines the size of this contribution.

Characteristics of Leakage

1. As $n\omega_0$ approaches ω, the term $(T/2)(\omega-n\omega_0)$ in the equation approaches 0:

$$\boldsymbol{Power\ (n)} = \frac{2\sin^2\left[\frac{T}{2}\left(\omega - n\omega_0\right)\right]}{\left[T\left(\omega - n\omega_0\right)\right]^2} = \frac{1}{2}\left[\frac{\sin\frac{T}{2}\left(\omega - n\omega_0\right)}{\frac{T}{2}\left(\omega - n\omega_0\right)}\right]^2$$

The sinc function has this property: $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

Thus, in the limit as $n\omega_0$ approaches ω, $\omega - n\omega_0$ approaches 0, and the sinc^2 function approaches 1.

$$\textit{Power (n)} = \frac{2\sin^2\left[\frac{T}{2}\left(\omega - n\omega_0\right)\right]}{\left[T\left(\omega - n\omega_0\right)\right]^2} = \frac{1}{2}\left[\frac{\sin\frac{T}{2}\left(\omega - n\omega_0\right)}{\frac{T}{2}\left(\omega - n\omega_0\right)}\right]^2$$

Therefore, in the limit as $n\omega_0$ approaches ω , the power at n approaches 0.5, which is the total power of x(t).

$$\boldsymbol{Power\ (n)} = \frac{2\sin^2\left[\frac{T}{2}\left(\omega - n\omega_0\right)\right]}{\left[T\left(\omega - n\omega_0\right)\right]^2} = \frac{1}{2}\left[\frac{\sin\frac{T}{2}\left(\omega - n\omega_0\right)}{\frac{T}{2}\left(\omega - n\omega_0\right)}\right]^2$$

2. As $n\omega_0$ moves away from ω, the denominator of the sinc^2 function in the equation gets larger, and the sinc^2 function gets smaller. This means that the further away $n\omega_0$ is from ω, the smaller is the contribution from the component at ω to the computed power at $n\omega_0$.

$$\textit{Power (n)} = \frac{2\sin^2\left[\frac{T}{2}\left(\omega - n\omega_0\right)\right]}{\left[T\left(\omega - n\omega_0\right)\right]^2} = \frac{1}{2}\left[\frac{\sin\frac{T}{2}\left(\omega - n\omega_0\right)}{\frac{T}{2}\left(\omega - n\omega_0\right)}\right]^2$$

3. When T is larger, the denominator of the sinc^2 function is larger, and the sinc^2 function changes more rapidly as $n\omega_0$ moves away from ω. This means that for larger T, the sinc^2 function is more compressed.

$$\textit{Power (n)} = \frac{2\sin^2\left[\frac{T}{2}\left(\omega - n\omega_0\right)\right]}{\left[T\left(\omega - n\omega_0\right)\right]^2} = \frac{1}{2}\left[\frac{\sin\frac{T}{2}\left(\omega - n\omega_0\right)}{\frac{T}{2}\left(\omega - n\omega_0\right)}\right]^2$$

An interpretation of the sinc^2 shape of the power spectrum comes from considering that a time series of finite length T may be thought of as the result of multiplying an infinitely long time series by a <u>rectangular time window</u> of length T (i.e. 1 within T and 0 elsewhere):

$$G_R(t) = 1 \text{ for } |t| <= T/2$$

$$= 0 \text{ for } |t| > T/2$$

(where the window extends from –T/2 to T/2)

The DFT of the finite-length time series is thus the DFT of the product of the time series and the time window.

The DFT of the time window $G_R(t)$ is a sinc function:

$$W_R = T\,\frac{\sin\left(\omega T/2\right)}{\omega T/2}$$

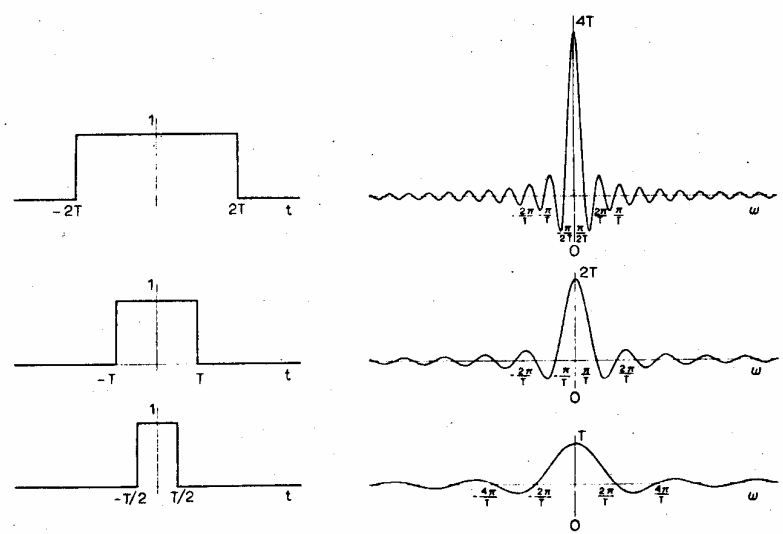


Fig. 7. *Time and spectral windows.* Left: time windows $G_R(t)$ (Eqn. G.2) of variable length; right: their corresponding spectral windows $W_R(\omega)$. Compression in the time domain is equivalent to expansion in the frequency domain.

The true DFT of the cosine is a single line at frequency ω.

The DFT of the product of the cosine and the rectangular window is the <u>convolution</u> of the true DFT of the cosine (the line) and the DFT of the time window (the sinc).

Definition of Convolution

Consider two time functions $f(\tau)$ and $k_1(\tau)$:

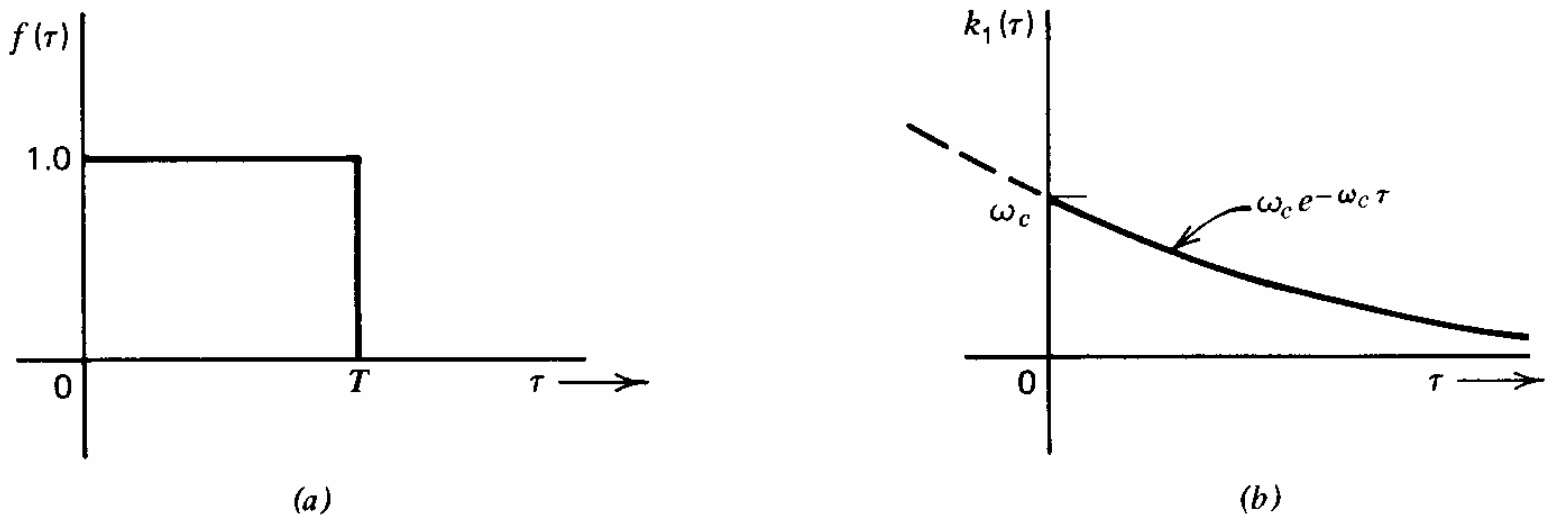


Figure 1-2. Plot of $f(\tau)$ and $k_1(\tau)$.

where τ represents time and t represents time lag.

Definition of Convolution

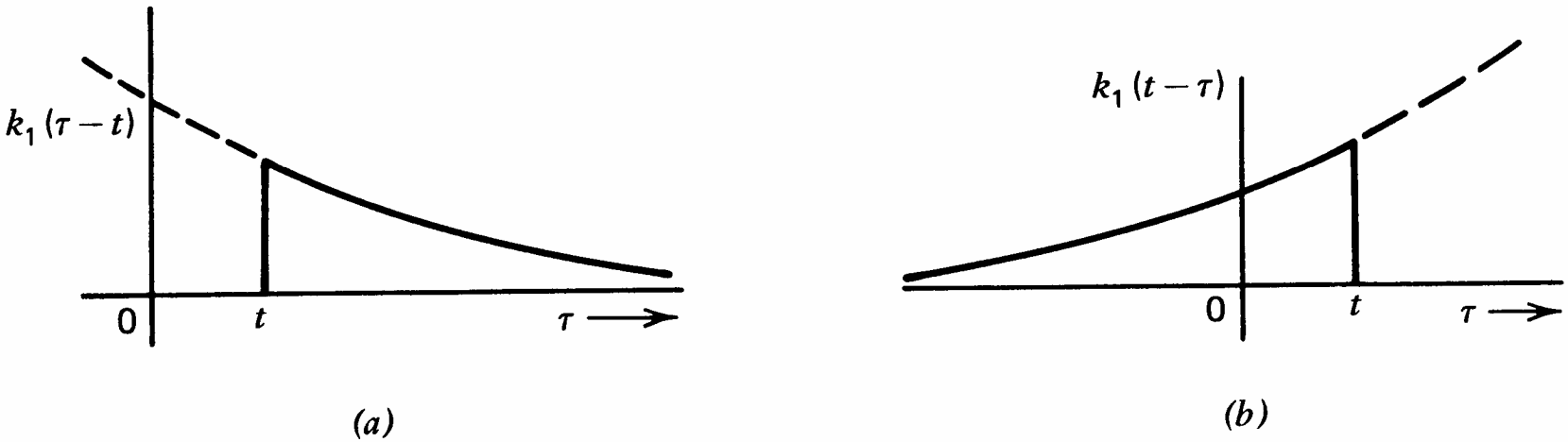


Figure 1-3. Plots of (a) $k_1(\tau - t)$ and (b) $k_1(t - \tau)$.

The figure on the left is $k_1(\tau)$ delayed by t.

The figure on the right is the same function reflected about the point $\tau = t$.

Definition of Convolution

The convolution of *f(t)* and $k_1(t)$ is given by:

$$u(t) = f(t) * k_1(t) \equiv \int_0^t f(\tau)k_1(t-\tau)d\tau$$

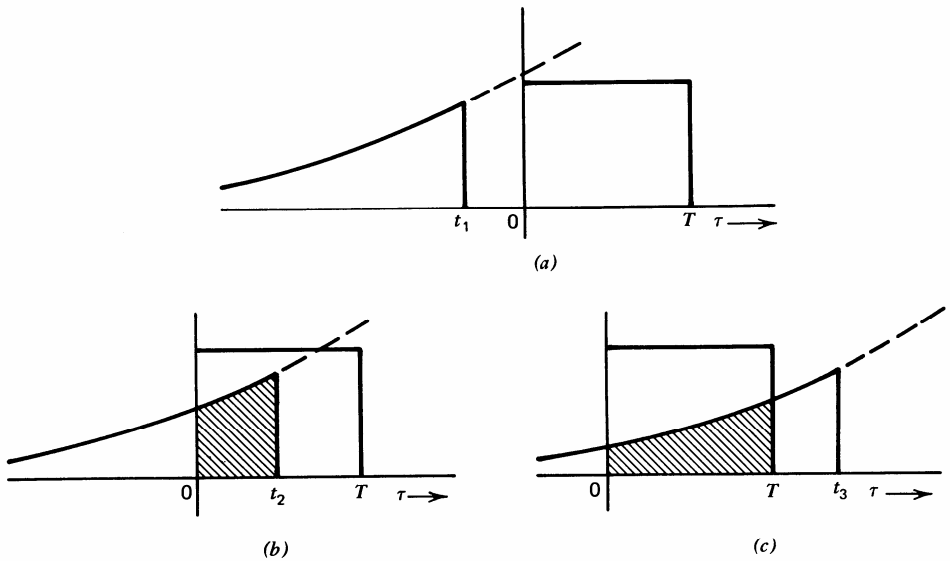


Figure 1-4. Convolution integral evaluation.

For each value of t, this integral equals the area under the function $f(\tau)k_1(t-\tau)$

from $\tau = 0$ to $\tau = t$. This area is indicated by the shading.

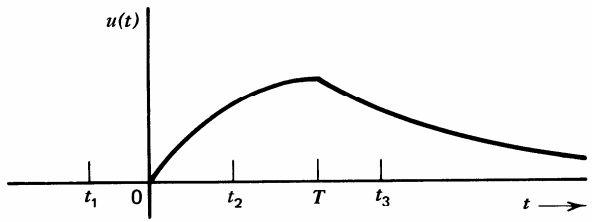


Figure 1-5. The convolution $u(t)$ of $f(t)$ and $k_1(t)$.

The Convolution Theorem

A general principle relating the time and frequency domains is expressed by the <u>convolution theorem</u>:

If X(t) and Y(t) are two real-valued time functions with Fourier transforms $Z_x(\omega)$ and $Z_Y(\omega)$, then:

(1) the Fourier transform of the convolution of X(t) and Y(t) is the product of $Z_x(\omega)$ and $Z_Y(\omega)$.

(2) the Fourier transform of the product of X(t) and Y(t) is the convolution of $Z_x(\omega)$ and $Z_Y(\omega)$.

The convolution theorem tells us that multiplication in the time domain is equivalent to convolution in the frequency domain, and convolution in the time domain is equivalent to multiplication in the frequency domain.

Thus, the DFT of the cosine segment is equivalent to the convolution of the DFT of the cosine (line spectrum) with the DFT of the time window (sinc function).

In general, this would apply to both the real and imaginary coefficients, but we only need to consider the real spectrum for a cosine function.

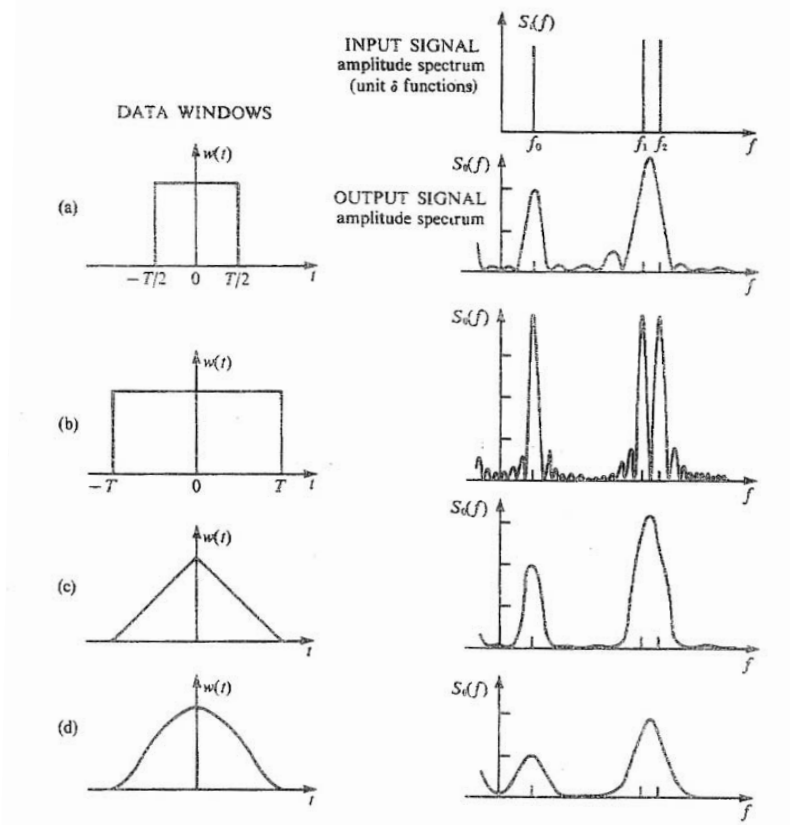


FIG. 2.10: Effect of data window shape and duration on signal spectra

To get the power spectrum of the cosine function, the real coefficient of the DFT is squared.
Thus, the power spectrum of the finite-length segment of the cosine function is equivalent to the convolution of the power spectrum of the cosine function with the sinc^2 function.

What to do about leakage

Leakage cannot be prevented.

However, its effect can be reduced by changing the shape of the time window in a way that reduces the side lobes of the spectral window.

Windowing

It can be shown that the problem of side lobes in the spectral window corresponding to the rectangular time window is due to the strong discontinuities at the edges.

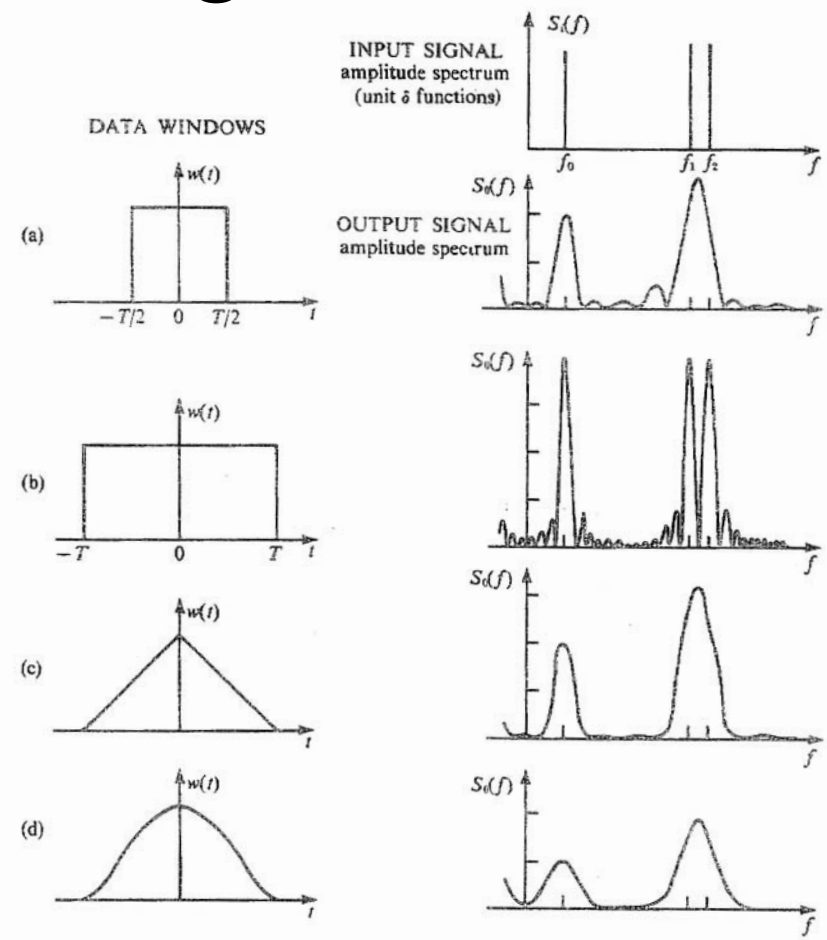


FIG. 2.10: Effect of data window shape and duration on signal spectra

Substituting a triangular window (c) (also called the Bartlett window) for the rectangular window is one way to reduce the side lobes, because it lessens the edge discontinuities. However, the triangular window is still discontinuous at the edges and the center.
A further modification is to use some form of bell-shaped time window (d). This shape is similar to the triangular window but smoothes out the discontinuities at the edges and the center.

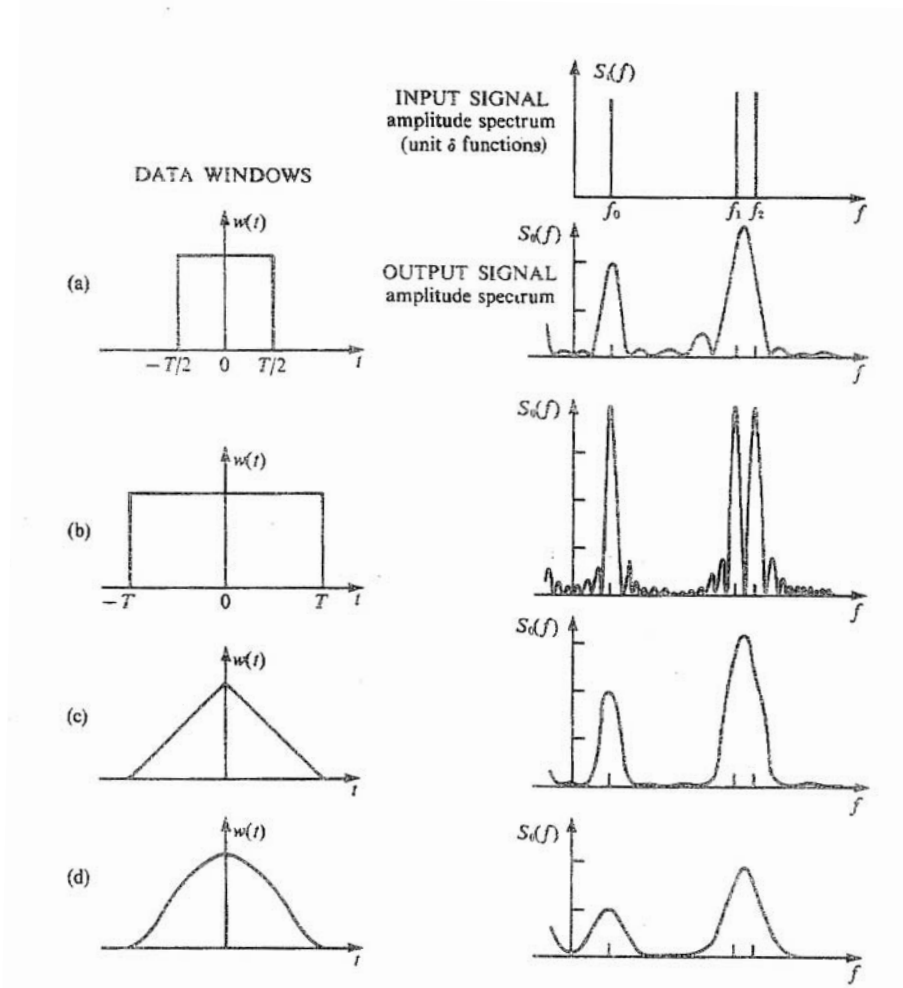


FIG. 2.10: Effect of data window shape and duration on signal spectra

Some popular windows

A popular bell-shaped window is the <u>Hanning window</u>:

$$G_H(k) = 0.5 - 0.5\cos\left(\frac{2\pi k}{N}\right)$$

where $X(k\Delta t)$ is multiplied by $G_H(k\Delta t)$ over the interval from k = 1 to N. The procedure of applying a Hanning window to data is called “Hanning”.

A slightly different procedure is called “Hamming”. The <u>Hamming window</u> is similar to the Hanning window with the first 0.5 replaced by 0.54 and the second 0.5 replaced by 0.46.

Detrending

A type of misrepresentation related to leakage is called trend. This results from the presence of frequency components lower than the fundamental frequency.

Power at these frequencies leaks into the Fourier components by $1/n^2$ (where n is the integer multiplier of the fundamental frequency.

To eliminate contamination of the spectrum by this type of leakage, trend should be estimated and removed before Fourier analysis, a process known as detrending.